J. Fluid Mech. (1983), *vol.* 127, *pp.* 497–506
Printed in Great Britain

On the application of the integral invariants and decay laws of vorticity distributions

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(Received 31 August 1981 and in revised form 28 June 1982)

Unsteady three-dimensional incompressible viscous flow fields induced by initial vorticity distributions are studied. Relevant invariants and decay laws of the moments of vorticity distributions are presented and shown to be useful in the numerical calculation of flow fields in two ways. First, the moments determine the leading terms of the far-field velocity, which can be employed as boundary conditions for the numerical calculation. Secondly, the deviations of the numerical results from the invariants and the decay laws can be used to measure the error of the numerical solution.

1. Introduction

We consider an incompressible viscous flow field induced by an initial vorticity distribution $\zeta(\mathbf{X})$ in three dimensions, where $\mathbf{X}$ denotes the position vector with Cartesian coordinates x_i, $i = 1$, 2 and 3. The velocity $\mathbf{V}(\mathbf{X}, t)$, the vorticity $\mathbf{\Omega}(\mathbf{X}, t)$ and the pressure p are governed by the differential equations

$$\mathbf{\Omega} = \nabla \times \mathbf{V}, \tag{1}$$

$$\nabla . \mathbf{V} = 0, \tag{2}$$

$$\frac{d}{dt}\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{V}, \tag{3}$$

where ρ and ν are the constant density and kinematic viscosity respectively. The initial condition at $t = 0$ is given by

$$\mathbf{\Omega}(\mathbf{X}, 0) = \zeta(\mathbf{X}). \tag{4}$$

We shall use u_i and ω_i to denote the components of $\mathbf{V}$ and $\mathbf{\Omega}$ respectively. The initial data ζ is of bounded support or decays exponentially in r, which denotes $|\mathbf{X}|$. Consequently, the vorticity decays exponentially in r for $t \geqslant 0$,

$$|\mathbf{\Omega}(\mathbf{X}, t)| = O(e^{-br}) \tag{5}$$

for some positive constant b, and the induced velocity field will be at rest at infinity, i.e.

$$|\mathbf{V}| \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \tag{6}$$

We note that (1)–(4) and (6) define an initial-value problem in an unbounded domain. However, numerical solutions of this problem can only be constructed for a bounded domain D, for which we have to impose appropriate boundary conditions and assess the error ϵ_b introduced by these conditions. The error ϵ_b will be related to the size R of the domain D. The error for the numerical solution is now composed of two parts ϵ_b and ϵ_d. The latter, ϵ_d, denotes the error of the finite-difference approximation, and is related to the grid size and the time step. For a given degree of accuracy, we shall then be able to specify the required grid size, time step and the size R of the domain D.

Although the condition (6) on $\mathbf{V}$ at infinity suggests that we impose the approximate condition

$$\mathbf{V} = 0 \tag{7}$$

on the boundary ∂D for the numerical solution, we cannot assess the error ϵ_b of the approximation in terms of the size R of the domain D. We therefore have to derive a far-field behaviour for $\mathbf{V}$, i.e. how $|\mathbf{V}| \to 0$ as $r \to \infty$. This is accomplished by making use of the far-field behaviour of vorticity, (5).

We introduce $\mathbf{\Omega}$ as a prime variable governed by the vorticity evolution equation derived from (3):

$$\mathbf{\Omega}_t + (\mathbf{V}\,.\,\nabla)\,\mathbf{\Omega} - (\mathbf{\Omega}\,.\,\nabla)\,\mathbf{V} = \nu\Delta\mathbf{\Omega}. \tag{8}$$

The velocity $\mathbf{V}$ is related to its vector potential $\mathbf{A}$ by the definition

$$\mathbf{V} = \nabla\times\mathbf{A}, \tag{9a}$$

$$\nabla\,.\,\mathbf{A} = 0. \tag{9b}$$

The vector potential $\mathbf{A}$ is then related to $\mathbf{\Omega}$ by the vector Poisson equation

$$\Delta\mathbf{A} = -\mathbf{\Omega}. \tag{10}$$

Equations (1)–(3) are now replaced by (8)–(10). The solution of (10) for a given vorticity distribution with boundary condition (6) is

$$\mathbf{A} = \frac{1}{4\pi}\iint_{-\infty}^{\infty} \frac{\mathbf{\Omega}(\mathbf{X}',t)}{|\mathbf{X}'-\mathbf{X}|}\,d\mathbf{X}', \tag{11}$$

where $d\mathbf{X}'$ denotes $dx_1'\,dx_2'\,dx_3'$. From (9a) we relate $\mathbf{V}$ to $\mathbf{\Omega}$, and (8) then becomes an integro-differential equation for $\mathbf{\Omega}$.

For the numerical solution of the problem in a finite domain D, we only have to impose the approximate boundary condition on $\mathbf{\Omega}$, i.e.

$$\mathbf{\Omega} = 0 \quad \text{on } \partial D. \tag{12}$$

The error of this approximation is

$$\epsilon_b = O(e^{-bR}),$$

which follows from the condition (5) on $\mathbf{\Omega}$. The error of the numerical solution of the integro-differential equation is now composed of three parts ϵ_b, ϵ_d and ϵ_D. The last one, ϵ_D, denotes the error of the approximation to the integral (11) by the integration over the finite domain D only. ϵ_D can be estimated by (5), and is found to be of the same order as ϵ_b, i.e. of exponential order in R. Numerical solutions of the integro-differential equation for vorticity were carried out for two-dimensional problems by Wu & Thompson (1973) and by Lo & Ting (1976).

Unfortunately, this scheme proves to be very inefficient in that the number of computational steps in evaluating the integral (11) for all N grid points in D is of order N^2. This is an order of magnitude larger than the number of computational steps for solving the original system of differential equations (8)–(10), which is of order $N \ln N$.

We arrive at the number $N \ln N$ by considering the numerical solution of (8)–(10) to consist of two parts: (i) the determination of the velocity field from a vorticity distribution by inverting the Poisson equation (10), and (ii) the evolution of vorticity from (8), which is the only equation involving a t-derivative. For each time increment the number of computational steps is dominated by the number needed to invert the Poisson equation, which is of the order $N \ln N$.

On the other hand, numerical solutions of the system of equations (8)–(10) require not only the far-field behaviour of $\mathbf{\Omega}$ but also that of $\mathbf{V}$. We shall demonstrate that the far-field behaviour of $\mathbf{V}$ can be derived from the far-field expansion of the Poisson integral. However, before doing this we make two remarks. First, we note that the Poisson integral could be used to define the boundary value of $\mathbf{V}$ on ∂D. This method is also inefficient in that the number of computational steps for generating the boundary data of $\mathbf{V}$ is of order $N^{\frac{5}{3}}$, which is still much larger than $N \ln N$. Secondly, we observe from the Poisson integral that

$$|\mathbf{A}| = O(r^{-1}), \tag{13a}$$

and hence

$$|\mathbf{V}| = O(r^{-2}). \tag{13b}$$

If we impose $\mathbf{V} = 0$ on ∂D, the error is of order R^{-2}. If we can prescribe more-accurate boundary data for $\mathbf{V}$, instead of $|\mathbf{V}| = 0$, we can maintain the same degree of accuracy while reducing R, i.e. the size of D, and hence reduce the total computational time by an order of magnitude. For example, if we require the error to be less than ϵ, R has to be of order $\epsilon^{-\frac{1}{2}}$ on account of (13b). If the error of the boundary data of $\mathbf{V}$ is reduced to $O(R^{-5})$, we need only R to order $\epsilon^{-\frac{1}{5}}$. Thereby we reduce the size of the domain by a factor of $\epsilon^{-\frac{3}{10}}$ and the total computational time by a factor of $\epsilon^{-\frac{9}{10}}$. We reach the latter conclusion by maintaining the same step sizes, that is, the same degree of accuracy for the finite-difference scheme.

To generate more-accurate boundary data for $\mathbf{V}$ or $\mathbf{A}$, we make use of the far-field behaviour of the Poisson integral (11), which is the expansion of

$$\mathbf{A}(\mathbf{X}, t) = \frac{1}{4\pi} \sum_{n=0}^{m} \mathbf{A}^{(n)}(\hat{\mathbf{r}}, t)\, r^{-n-1} + O(r^{-m-2}), \tag{14}$$

where

$$\mathbf{A}^{(n)}(\hat{\mathbf{r}}, t) = \iiint_{-\infty}^{\infty} \mathbf{\Omega}(\mathbf{X}', t)\, (r')^n P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')\, d\mathbf{X}'.$$

Here $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ denote unit vectors in the directions of $\mathbf{X}$ and $\mathbf{X}'$ respectively. P_n is a Legendre polynomial, and $(r')^n P_n$ is a homogeneous polynomial in x_i' of degree n. Consequently $r^n \mathbf{A}^{(n)}$ is a homogeneous polynomial in x_i of degree n, and its coefficients are the nth moments of vorticity. The first three terms of (14) are

$$\mathbf{A}^{(0)} = \sum_{i=1}^{3} \langle \omega_i \rangle \hat{\tau}_i, \tag{15}$$

$$\mathbf{A}^{(1)} = \sum_{i=1}^{3} \sum_{j=1}^{3} \langle x_j \omega_i \rangle \lambda_j \hat{\tau}_i, \tag{16}$$

$$\mathbf{A}^{(2)} = \sum_{i=1}^{3} \sum_{j,k=1}^{3} \langle x_j x_k \omega_i \rangle \tfrac{1}{2}\{3\lambda_j \lambda_k - \delta_{jk}\} \hat{\boldsymbol{\tau}}_i, \tag{17}$$

where λ_i are the components of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\tau}}_i$ are the unit vectors parallel to the coordinate axes, and $\langle\ \rangle$ denotes the volume integral over the entire space.

We note that the coefficient of the nth term in the far-field expansion (14) is of the form

$$\mathbf{A}^{(n)}(\hat{r}, t) = \sum_{i=1}^{3} \sum_{l=1}^{N} M_{li}^{(n)}(t)\, G_{li}^{(n)}(\hat{\mathbf{r}})\, \hat{\boldsymbol{\tau}}_i,$$

where $3N$ is the number of nth moments of components of $\mathbf{\Omega}$ and is equal to $\frac{3}{2}(n+2)(n+1)$. Therefore we can say that the nth term in (14) has that many components $G_{li}^{(n)}$, and the nth moment $\mathrm{M}_{li}^{(n)}$ defines the strength of that component. It is therefore of importance to relate the moments, or linear combinations of them, directly to their initial values before solving the initial-value problem. Such relationships will be useful in getting the far-field velocity. They can also be used to verify the accuracy of the computational code for the problem and to measure the error due to the finiteness of the domain.

Poincaré (1893) obtained integral invariants and a decay law for the moments of a two-dimensional vorticity distribution. For the three-dimensional problem, invariants of linear combinations of components of the nth moments (referred to as 'the nth symmetrical moments of vorticity') were derived by Truesdell (1951) from the divergence-free condition on vorticity and its far-field behaviour. Additional results were obtained by Moreau (1948, 1949) regarding the first and the second moments from the vorticity-diffusion equation. Unified proofs of these results from a formula of the type called 'vorticity divergence formula' and a systematized search of all possible formulas of the same type were presented by Howard (1957).

In §2, we will list the results of Truesdell, Moreau and Howard which will be needed in §3 to define the far-field behaviour of the velocity field. The equations defining the boundary data of the numerical solutions are presented for three-dimensional problems and then for the special cases of axisymmetrical and two-dimensional problems.

For the sake of completeness, outlines for the derivations of these formulas listed in §2 are presented in the appendix. Systematic and thorough presentations and derivations of them can be found in Howard (1957) and Truesdell (1954).

2. Integral invariants and decay laws

In this section we list those relationships which are pertinent to the derivation of the far-field behaviour of the velocity.

From the definition (1) of $\mathbf{\Omega}$, we see that it is divergence-free:

$$\nabla . \mathbf{\Omega} = 0. \tag{18}$$

Using the above and the far-field behaviour of $\mathbf{\Omega}$, it was shown that the nth coaxial moment along an axis parallel to a vector $\mathbf{B}$ should vanish (see the appendix and Truesdell 1951):

$$I^{(n)}(b_1, b_2, b_3, t) = \iiint_{-\infty}^{\infty} [\mathbf{B}.\mathbf{X}]^n \mathbf{B}.\mathbf{\Omega}(\mathbf{X}, t)\, d\mathbf{X} = 0 \tag{19}$$

for $t \geqslant 0$, $n = 0, 1, 2, \ldots$, and for all b_i, which are the components of $\mathbf{B}$. Since $I^{(n)}(b_i)$ is a homogeneous polynomial in b_i of degree $n+1$, (19) holds only if all the coefficients in the polynomial are equal to zero. There are $\frac{1}{2}(n+3)(n+2)$ coefficients, which are linear combinations of nth moments of vorticity. Hence we have $\frac{1}{2}(n+3)(n+2)$ consistency conditions for the $\frac{3}{2}(n+2)(n+1)$ components of the nth moments. The results of this are stated for $n = 0$, 1 and 2 in the following.

For $n = 0$ we have three conditions:

$$\langle \omega_i \rangle = 0 \quad (i = 1, 2, 3). \tag{20}$$

Equation (20) states that the total vorticity has to be zero for all time, i.e.

$$\langle \mathbf{\Omega} \rangle = 0.$$

As a result, (15) becomes $\mathbf{A}^{(0)} = 0$, and (14) says that

$$\mathbf{A}(\mathbf{X},t) = \frac{1}{4\pi}\mathbf{A}^{(1)}(\hat{\mathbf{r}},t)\,r^{-2}+O(r^{-3}). \tag{21}$$

Consequently we obtain the far-field behaviour of $\mathbf{V}$:

$$|\mathbf{V}| = |\nabla\times\mathbf{A}| = O(r^{-3}). \tag{22}$$

For $n = 1$ we have six conditions:

$$\langle x_i\,\omega_j\rangle+\langle x_j\,\omega_i\rangle = 0 \quad (i,j = 1,2,3;\quad j\geqslant i). \tag{23}$$

Equation (23) says that the 3×3 matrix of the first moments $\langle \mathbf{X}\mathbf{\Omega}^*\rangle$ is skew-symmetric. Here we represent a vector by a column matrix and its transpose by $(*)$.

For $n = 2$ we have ten conditions for the eighteen second moments:

$$\langle x_i^2\,\omega_i\rangle = 0 \quad (i = 1,2,3), \tag{24a}$$

$$\langle x_i^2\,\omega_j\rangle +2\langle x_i\,x_j\,\omega_i\rangle = 0 \quad (i,j = 1,2,3;\quad i\neq j), \tag{24b}$$

$$\langle x_1x_2\omega_3\rangle+\langle x_1\omega_2x_3\rangle+\langle \omega_1x_2x_3\rangle = 0. \tag{24c}$$

In addition to those consistency conditions, only a finite number of integral invariants were obtained by Moreau (1948, 1949) for the vorticity evolution equation (8). They consist of three equations for $n = 1$, three for $n = 2$, and none for $n\geqslant 3$, and are accounted for in the following way.

For $n = 1$, we can show from (8) that the matrix $\langle \mathbf{X}\mathbf{\Omega}^*\rangle$ is time invariant. Since the matrix is skew-symmetric, (23), we have only three non-trivial time invariants. We choose the following three:

$$\langle x_i\,\omega_j\rangle = \langle x_i\,\zeta_j\rangle = C_k \quad (k = 1,2,3;\quad i\neq j\neq k;\quad i<j). \tag{25}$$

For $t\geqslant 0$ the second moments are defined by the three constants C_k:

$$\langle \mathbf{X}\mathbf{\Omega}^*\rangle = \begin{bmatrix} 0 & C_3 & C_2\\ -C & 0 & C_1\\ -C_2 & -C_1 & 0\end{bmatrix}. \tag{26}$$

For $n = 2$, we have, in addition to the ten consistency conditions (24), three non-trivial invariants. They are

$$\langle r^2\,\omega_i\rangle = \langle r^2\,\zeta_i\rangle = D_i \quad (i = 1,2,3), \tag{27}$$

and are equivalent to $\langle r^2\mathbf{\Omega}\rangle = \boldsymbol{D}$. The latter says that the polar moment of vorticity with respect to the origin is time-invariant.

Altogether we now have thirteen conditions (24) and (27) for the eighteen second moments. The remaining five linearly independent integrals can be chosen as follows:

$$G_k(t) = \langle (x_i^2-x_j^2)\,\omega_k\rangle \quad (k = 1,2,3), \tag{28}$$

$$H_k(t) = \langle 2x_i\,x_j\,\omega_k-x_i\,x_k\,\omega_j-x_k\,x_j\,\omega_i\rangle \quad (k = 1,2), \tag{29}$$

where $i\neq k\neq j$ and $i<j$. We note that $H_1+H_2+H_3\equiv 0$. The above five integrals are of course time-dependent and can be defined approximately at each instant of time by numerical integrations over the finite domain D.

For $n\geqslant 3$ we have only $\frac{1}{2}(n+3)(n+2)$ consistency conditions from (19), for the $\frac{3}{2}(n+2)(n+1)$ components of the nth moments. To define all the nth moments we have to compute $n(n+2)$ of them by numerical integrations over D.

The relationships listed in this section will be employed in §3 to define the boundary data for the vector velocity potential. Furthermore we note that these relationships are valid for integrals over the entire space, while the corresponding integrals of the numerical solutions are evaluated over the finite domain D only. The deviations of the numerical results from those integral relationships can therefore be used to serve the purpose of measuring the overall effects of the size of the domain D on the accuracy of the numerical solutions.

For the same purpose, we shall include one additional integral relationship which expresses the rate of decay of the total energy T. From the integral of the energy equation we obtain

$$T'(t) = -\langle\Phi\rangle, \tag{30}$$

where

$$T = \tfrac{1}{2}\langle \mathbf{V}\,.\,\mathbf{V}\rangle = \tfrac{1}{2}\langle \mathbf{A}\,.\,\mathbf{\Omega}\rangle,$$

$$\langle\Phi\rangle = \tfrac{1}{2}\nu\left\langle \sum_{i=1}^{3} (\nabla v_i)^2\right\rangle = -\tfrac{1}{2}\nu\langle \mathbf{V}\,.\,(\nabla\times\mathbf{\Omega})\rangle.$$

In deriving (30) we integrate by parts and make use of the far-field behaviour of $\mathbf{V}$, $\mathbf{A}$ and $\mathbf{\Omega}$ repeatedly. In particular, we note from (16), (21), (22) and (26) that $\mathbf{V}_t$ is $O(r^{-4})$, while $\mathbf{V}$ is $O(r^{-3})$. Consequently, we find that p is $O(r^{-3})$. This condition is needed to show that the work $\langle \mathbf{V}\,.\,\nabla p\rangle$ done due to pressure vanishes. Equation (30) then becomes a decay law for $\langle \mathbf{A}\,.\,\mathbf{\Omega}\rangle$:

$$\langle \mathbf{A}\,.\,\mathbf{\Omega}\rangle = \langle \mathbf{A}\,.\,\zeta\rangle_{t=0} + \nu\int_0^t \langle \mathbf{V}\,.\,\nabla\times\mathbf{\Omega}\rangle\,dt. \tag{31}$$

3. Far-field conditions

When the vorticity decays expontially in the far field the vector velocity potential can be represented as a power series in r^{-1}:

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{4\pi r^2}\mathbf{A}^{(1)}(\hat{\mathbf{r}}) + \frac{1}{4\pi r^3}\mathbf{A}^{(2)}(\hat{\mathbf{r}},t) + O(r^{-4}).$$

The leading term of the series in (14), which is of order r^{-1}, is dropped because of the consistency condition (20). The term that is of order r^{-2} represents the contributions of 'dipoles' at the origin. Their strengths are defined by the nine components of $\langle \mathbf{X}\mathbf{\Omega}^*\rangle$, the matrix of the first moments. The matrix is time-invariant and is defined in (26) by three constants C_k, which are specified by the initial data in (25). Consequently (16) and (26) yield

$$\mathbf{A}^{(1)}(\hat{\mathbf{r}}) = \sum_{k=1}^{3} C_k(\lambda_i\hat{\mathbf{t}}_j - \lambda_j\hat{\mathbf{t}}_i), \tag{33}$$

where $i \neq k \neq j$ with $i<j$. Recall that $\hat{\mathbf{t}}_k$, $k = 1, 2, 3$, are the unit vectors along the coordinates axes and λ_k are the direction cosines of $\mathbf{X}$ and hence the components of $\hat{\mathbf{r}}$. We note that $\mathbf{A}^{(1)}$, the coefficient of r^{-2} in (32), is independent of t.

The second term in (32) represents the contribution of 'quadrupoles' at the origin. Their strengths are defined in (17) by the eighteen components of the second moments $\langle x_i x_j \omega_k\rangle$. By using the ten consistency conditions (24a–c) and the integral invariants (27), the coefficient $\mathbf{A}^{(2)}$ can be written as

$$\begin{aligned}\mathbf{A}^{(2)}(\hat{\mathbf{r}},t) = {}& \frac{3}{4}\sum_{k=1}^{3} D_k[(\lambda_i^2+\lambda_j^2-\tfrac{2}{3})\hat{\mathbf{t}}_k - \lambda_k(\lambda_i\hat{\mathbf{t}}_i+\lambda_j\hat{\mathbf{t}}_j)] \\ & + \frac{3}{4}\sum_{k=1}^{3} G_k(t)\,[(\lambda_i^2-\lambda_j^2)\hat{\mathbf{t}}_k - \lambda_k(\lambda_i\hat{\mathbf{t}}_i-\lambda_j\hat{\mathbf{t}}_j)] + \sum_{k=1}^{3} H_k(t)\,\lambda_i\lambda_j\hat{\mathbf{t}}_k \\ & (i\neq j\neq k;\ i<j). \end{aligned}\tag{34}$$

The three constants D_k are defined by the initial data in (27). The last two terms in (34) involve the time-dependent second moments $G_k(t)$ and $H_k(t)$. At each instant we have to evaluate only those five integrals defined in (28) and (29) for $G_k(t)$ and $H_k(t)$ and will then be able to define $\mathbf{A}$ to order r^{-3}.

If we construct the numerical solution of the Navier–Stokes equation for the evolution of vorticity, and then the Poisson equation for the vector velocity potential, we can use (12) and (32) as the boundary conditions for $\mathbf{\Omega}$ and $\mathbf{A}$ respectively. The error due to the boundary data is dominated by that for $\mathbf{A}$, and hence is $O(R^{-4})$. By equating this error to that of the finite-difference equations we can relate the step sizes to the size of the domain.

In principle we can obtain a better estimate of $\mathbf{A}$ in the far field by the addition of the terms involving third and higher moments. However, improvement may not materialize because small numerical errors in the vorticity distribution in the far field will be amplified in those $n(n+2)$ components of higher moments which can only be evaluated approximately by numerical integration over the domain D.

In §3.1 and 3.2 we shall reduce the results of the general three-dimensional case to the axisymmetric one and to the two-dimensional case with appropriate modifications.

3.1. *Axisymmetric flow*

For the special case of an axisymmetric flow in which the vortex lines are coaxial circles, the vorticity vector can be expressed in terms of its circumferential components Θ,

$$\mathbf{\Omega} = \Theta(\sigma, x_3, t)\,\hat{\boldsymbol{\theta}} = [-\sin\theta\hat{\mathbf{t}}_1 + \cos\theta\hat{\mathbf{t}}_2]\,\Theta, \tag{35}$$

where σ, θ, x_3 denote the cylindrical coordinates. It is clear that $\mathbf{\Omega}$ is divergence-free; therefore all the consistency conditions are fulfilled automatically. For the integral invariants of the first moments (26) the non-trivial ones are $\langle x_1\,\omega_2\rangle$ and $\langle x_2\,\omega_1\rangle$. We thus have

$$\langle x_1\,\omega_2\rangle = -\langle x_2\,\omega_1\rangle = \int_{-\infty}^{\infty}\int_0^{\infty} \pi\sigma^2\Theta(\sigma, x_3, t)\,d\sigma\,dx_3 = \text{constant} = C_3. \tag{36}$$

We note that a circular vortex filament, with cross-sectional area $d\sigma\,dx_3$ and strength $\Theta\,d\sigma\,dx_3$, is equivalent to a uniform distribution of doublets with the same strength over the circular disk spanned by the filament. Consequently (36) expresses the conservation of the total doublet strength. The use of this conservation law to check the accuracy of the numerical solutions was suggested by Ting (1981).

For the second moments, only two of the eighteen components are non-trivial. Those two, $\langle x_3\,x_1\,\omega_2\rangle$ and $\langle x_2\,x_3\,\omega_1\rangle$, are related to one integral, which represents the first moment of the equivalent double distribution with respect to x_3, and are given by

$$\langle x_3\,x_1\,\omega_2\rangle = \tfrac{1}{3}H_2(t) = -\langle x_2\,x_3\,\omega_1\rangle = -\tfrac{1}{3}H_1(t) = \int_{-\infty}^{\infty}\int_0^{\infty} X_3[\pi\sigma^2\Theta]\,d\sigma\,dx_3. \tag{37}$$

For the thirty components of third moments, only six are non-trivial. They are related to two integrals as follows:

$$\langle x_2^2\,x_1\,\omega_2\rangle = -\langle x_1^2\,x_2\,\omega_1\rangle = \tfrac{1}{3}\langle x_1^3\,\omega_2\rangle = -\tfrac{1}{3}\langle x_2^3\,\omega_1\rangle$$

$$= I_1(t) = \frac{1}{4}\int_{-\infty}^{\infty}\int_0^{\infty} \sigma^2[\pi\sigma^2\Theta]\,d\sigma\,dx_3, \tag{38a}$$

$$\langle x_3^2\,x_1\,\omega_2\rangle = -\langle x_3^2\,x_2\,\omega_1\rangle = I_2(t) = -\int_{-\infty}^{\infty}\int_0^{\infty} z^2[\pi\sigma^2\Theta]\,d\sigma\,dx_3. \tag{38b}$$

Using these results, we finally obtain the far-field vector potential for an axisymmetric flow field:

$$\mathbf{A} = \hat{\theta}\psi(\sigma, x_3, t),$$

with

$$\psi = \frac{\sigma}{4\pi r}\left\{C_3 r^{-2} + H_2(t)\frac{x_3}{r}r^{-3} + \tfrac{3}{2}r^{-4}\left[\left(\frac{5\sigma^2}{r^2}-4\right)I_1(t) + \left(\frac{5x_3^2}{r^2}-1\right)I_2(t)\right] + O(r^{-5})\right\}. \quad (39)$$

Here ψ can be identified as the stream function for the axisymmetric flow.

When the boundary data for ψ are defined by the three leading terms of (39) instead of $\psi = 0$, the error is reduced from order R^{-2} to R^{-5}. Consequently the more-accurate boundary data allow us to reduce the size of the domain and the computational time by an order of magnitude. This fact is demonstrated by Liu & Ting (1982) in their study of the self-merging of a vortex ring when its vortical core radius is comparable to the radius of the ring.

3.2. *Two-dimensional flow*

For a two-dimensional flow in the (x_1, x_2)-plane, we use (σ, θ) to denote the polar coordinates and define the far field by $\sigma \gg 1$. The vorticity vector and the vector velocity components have only one non-trivial component:

$$\boldsymbol{\Omega} = \omega_3(x_1, x_2, t)\,\hat{\mathbf{t}}_3, \quad \mathbf{A} = \psi(x_1, x_2, t)\,\hat{\mathbf{t}}_3, \quad (40)$$

where ψ is the stream function. When the vorticity decays exponentially in σ we can derive the far-field behaviour of $\mathbf{V}$ from the Poisson integral for the stream function. They are

$$|\omega_3| = O(e^{-p\sigma}), \quad |\mathbf{V}| = O(\sigma^{-1}), \quad (41\,a, b)$$

where p is a positive constant.

Since the two-dimensional vorticity distribution $\omega_3\,\hat{\mathbf{t}}_3$ is always divergence-free, there will be no consistency conditions. The integral invariants for the total vorticity and the first moments as well as a decay law for the second polar moment were obtained by Poincaré (1893). They are

$$\langle\omega_3\rangle = \Gamma, \quad \langle x_2\,\omega_3\rangle = C_1, \quad \langle x_1\,\omega_3\rangle = C_2, \quad \langle\sigma^2\,\omega_3\rangle = 4\nu\Gamma t + D_3, \quad (42)$$

where Γ, C_1, C_2 and D_3 are constants defined by the initial data. They are $\Gamma = \langle\zeta_3\rangle$, $C_1 = \langle x_2\,\zeta_3\rangle$, $C_2 = \langle x_1\,\zeta_3\rangle$ and $D_3 = \langle\sigma^2\,\zeta_3\rangle$.

For the second moments there are three components. We know only the polar moment, and choose the other two components to be

$$G_3(t) = \langle(x_1^2 - x_2^2)\,\omega_3\rangle, \quad (43a)$$

$$H_3(t) = \langle 2x_1\,x_2\,\omega_3\rangle. \quad (43b)$$

They have to be evaluated by numerical integration of the instantaneous vorticity distribution over the finite domain D.

Using (42) and (43) we obtain the far-field behaviour of the stream function:

$$\psi(\sigma, \theta, t) = \frac{\Gamma}{2\pi}\ln\sigma - \frac{1}{2\pi\sigma}[C_1\sin\theta + C_2\cos\theta] - \frac{1}{2\pi\sigma^2}[G_3(t)\cos 2\theta + H_3(t)\sin 2\theta] + O(\sigma^{-3}). \quad (44)$$

When $\Gamma \neq 0$ we can use (42) to show that the 'centre of gravity' of the vorticity distribution is stationary. We can then choose the centre of gravity as the origin and obtain $C_1 = 0$, $C_2 = 0$. Then, from the far-field behaviour of ψ in (44), we see a vortex and two quadrupoles located at the origin. Equation (44) is employed to define the boundary data by Weston & Liu (1982) in the study of the roll-up of vortex wakes. They show that their numerical scheme is much more efficient than the scheme of Steger & Kutler (1977), who used $\psi = 0$ as the boundary condition.

The author wishes to acknowledge Prof. David Stickler for valuable discussions and Prof. Steve Childress for pointing out the papers of Howard and Truesdell. This research is supported by an ONR contract.

Appendix. Derivations of formulas in §2

Formulas of the type (19) can be derived from the divergence-free condition of $\mathbf{\Omega}$ and its far-field behaviour in two steps. We will show than an nth 'coaxial' moment of $\mathbf{\Omega}$ should vanish. Those formulas of the type (19) will then follow from the fact that the direction of the axis can be arbitrarily assigned.

For the first step we note that the nth coaxial moment about the x_3 axis is

$$\iiint_{-\infty}^{\infty} x_3^n\, \omega_3\, d\mathbf{X} = \int_{-\infty}^{\infty} dx_3\, x_3^n \left[\iint_{-\infty}^{\infty} \omega_3\, dx_1\, dx_2 \right].$$

From the divergence theorem the integral over the plane of constant x_3, which can be written as $\iint_{-\infty}^{\infty} \mathbf{\Omega} \,.\, \hat{\mathbf{i}}_3\, dx_1\, dx_2$, is equal to a surface integral of $\mathbf{\Omega} \,.\, \hat{\mathbf{n}}$ over a hemisphere with radius $R \rightarrow \infty$. Using the far-field behaviour of $\mathbf{\Omega}$ we obtain the result for the coaxial moment about the x_3-axis:

$$\iiint_{-\infty}^{\infty} (\mathbf{X} \,.\, \hat{\mathbf{i}}_3)^n\, \mathbf{\Omega} \,.\, \hat{\mathbf{i}}_3\, dX = 0, \tag{A 1}$$

This statement is valid for an axis along any direction $\mathbf{B}$ with components b_i. Equation (A 1) becomes

$$\iiint_{-\infty}^{\infty} (\mathbf{X} \,.\, \mathbf{B})^n\, \mathbf{\Omega} \,.\, \mathbf{B}\, d\mathbf{X} = 0, \tag{A 2}$$

which is (19).

To derive the temporal relationship for the moments of vorticity, we apply the curl operator to (3) in its conservation form and obtain the equation for the temporal variation of vorticity:

$$\mathbf{\Omega}_t = -\nabla \times [\nabla \,.\, (\mathbf{VV})] + \nu \Delta \mathbf{\Omega}. \tag{A 3}$$

Each term on the right-hand side of the equation contains two differentiations with respect to the space variables. We then carry out the following four steps.

(i) We multiply both sides of (A 3) by $x_1^{n_1}\, x_2^{n_2}\, x_3^{n_3}$ and integrate both sides over the entire space. The left side becomes the rate of change on an nth moment, where n is the sum of the integers n_1, n_2 and n_3.

(ii) For each term on the right-hand side of the equation, we integrate by parts twice, to remove the spatial derivatives, yielding boundary terms and an $(n-2)$th moment of the vorticity or of a product of velocity components.

(iii) The far-field behaviour of vorticity and velocity shows that the boundary terms vanish, and that the integrals of the $(n-2)$th moments of the products of velocity components exist. This step can be carried out for $n < 4$ under condition (22).

(iv) For $n \geqslant 2$ we need one more step because the right-hand side will contain the integrals of the moments of the products of velocity components, which are unknown prior to the solution of the initial-value problem. We seek linear combinations of nth moments to eliminate terms involving products of velocity components.

Carrying out these four steps, we arrive at (25) and (27).

REFERENCES

HOWARD, L. 1957 *Arch. Rat. Mech. Anal.* **1**, 113–138.

LIU, C. H. & TING, L. 1982 Numerical solution of viscous flow in unbounded fluid. In *Proc. 8th Int. Conf. on Numerical Methods in Fluid Dynamics, Aachen, Germany*, 28 *June*–2 *July* (ed. E. Krause). Lecture Notes in Physics. Springer. (To appear.)

LO, R. K. C. & TING, L. 1976 *Phys. Fluids* **19**, 912–913.

MOREAU, J. J. 1948 *C.R. Acad. Sci. Paris* **226**, 1420–1422.

MOREAU, J. J. 1949 *C.R. Acad. Sci. Paris* **229**, 100–102.

POINCARÉ, H. 1893 *Théorie des Tourbillons* (ed. G. Carré), chap. IV. Deslis Frères.

STEGER, J. L. & KUTLER, P. 1977 *A.I.A.A. J.* **15**, 581–590.

TING, L. 1981 In *Advances in Fluid Mechanics* (ed. E. Krause). Lecture Notes in Physics, vol. 148, pp. 67–105. Springer.

TRUESDELL, C. 1951 *Can. J. Math.* **3**, 69–86.

TRUESDELL, C. 1954 *The Kinematics of Vorticity.* Indiana University Press.

WESTON, R. P. & LIU, C. H. 1982 Approximate boundary condition procedure for the two-dimensional numerical solution of vortex wakes. *A.I.A.A./A.S.M.E. 3rd Joint Thermo-Physics, Fluids, Plasma and Heat Transfer Conf. June, St Louis, MI, Paper* no. 82-0951.

WU, J. C. & THOMPSON, J. F. 1973 *Comp. Fluids* **1**, 197–215.